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Centrifugal instability of a Stokes layer: subharmonic destabilization of the Taylor vortex mode

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The centrifugal instability of a Stokes layer has been investigated by Seminara & Hall (1976, 1977). It was found that the Stokes layer on a torsionally oscillating circular cylinder is unstable to perturbations periodic along the axis of the cylinder when the Taylor number for the flow exceeds a certain critical value. The weakly nonlinear theory given by Seminara & Hall showed that, if nonlinear effects are considered, at this Taylor number a stable axially periodic equilibrium flow bifurcates from the basic circumferential flow. It is known experimentally that this equilibrium flow becomes unstable to disturbances having a longer axial wavelength at a second critical Taylor number about 10 % greater than the first critical value. Moreover it is known that, in the initial stages of this destabilization, a mode having twice the axial wavelength of the fundamental is present. In this paper we investigate the linear stability of the bifurcating solution to such a subharmonic mode. An approximate solution of the linear stability problem shows that the subharmonic becomes unstable at a Taylor number remarkably close to the experimentally measured second critical Taylor number.

1. Introduction

In this paper we are concerned with the stability of a Stokes layer on an infinitely long torsionally oscillating circular cylinder. We shall first give a brief account of some experimental results which are relevant to our investigation.

Seminara & Hall (1976, 1977), hereafter referred to as I and II respectively, described the results of a qualitative experimental investigation of the flow set up by a long circular cylinder oscillating torsionally about its axis. The cylinder used was such that its radius was large compared to the boundary layer set up by the motion of the cylinder. It was found that the Stokes layer is stable until the Taylor number for the flow reaches a critical value of about 210. At this stage a Taylor vortex equilibrium flow develops when the Taylor number is increased. This flow is periodic in time and along the axis of the cylinder. However, only the radial and axial velocity components have steady terms in their Fourier decomposition. If the Taylor number is increased further then, at a Taylor number of about 262, the equilibrium flow described above becomes unstable. At this Taylor number neighbouring vortices of the Taylor vortex flow appear to interact with each other to produce bigger vortices. The crude flow visualization method used in I was not good enough to ascertain the development of the flow after the initial interaction between neighbouring vortices. However, it was possible to see that, at some stage in the development of the flow after the second instability, the azimuthal velocity of the fluid acquired a steady component. It was not possible to tell with any certainty whether the new flow was dependent on the circumferential angle θ.

We now turn to the more recent quantitative experimental investigation of the problem by Donnelly & Park (1980). The latter authors were able to investigate in detail the development of the flow after the onset of the second instability. They also found that the first sign of this instability is the interaction of neighbouring cells in the Taylor vortex flow to produce bigger cells. The larger vortices then interact again to produce still bigger vortices. This process then continues but in a short time there is no apparent axial periodic structure to the flow. There is certainly no simple equilibrium configuration reached at the end of this process. At some stage after the initial 'doubling up' of the vortices the flow becomes θ dependent. In some cases Donnelly & Park observed that the equilibrium Taylor vortex flow was modulated along the axis of the cylinder on a length scale of the same order as the radius of the cylinder. In this case the onset of the second instability always occurred at the axial positions where the modulation had its maximum amplitude.

We shall now briefly describe the theoretical progress which has been made towards an understanding of the flow described above. In I the linear stability of a thin Stokes layer on a torsionally oscillating cylinder was investigated. It was found that the Stokes layer is unstable to centrifugal instabilities for Taylor numbers greater than 232·5. The weakly nonlinear theory given in II showed that at this Taylor number a stable equilibrium Taylor vortex flow bifurcates from the basic flow. The manner in which this bifurcation is altered by allowing the cylinder to be slightly wavy was discussed in detail by Duck (1979). There seems little doubt that the bifurcating equilibrium flow discussed in II is that observed experimentally before the second instability. Seminara (1979) has investigated the stability of the Stokes layer to θ-dependent perturbations. Such modes necessarily have azimuthal velocity components with steady terms in their Fourier decomposition. These modes were found to be more stable than the axisymmetric modes discussed in I. However, the axial wavelengths of these modes were found to be smaller than that of the most dangerous axisymmetric mode so that their importance in the initial stages of the second instability is perhaps not crucial. With this in mind we shall here study the linear stability of the stable bifurcating Taylor vortex flow to an axisymmetric mode having axial wavelength twice that of the basic cellular structure. The procedure adopted in the rest of the paper is as follows: In § 2 we formulate the linear stability problem which determines the stability of the equilibrium Taylor vortex flow to subharmonic perturbations. In § 3 we obtain an approximate solution of this problem whilst in § 4 we discuss the results of § 3 and their relevance to the experimental observations described above.

2. The linear stability problem for the subharmonic mode

We consider a circular cylinder of radius R oscillating torsionally about its axis in an unbounded viscous fluid of kinematic viscosity ν with angular velocity $\Delta\omega R^{-1}\cos\omega t$. This motion generates a boundary layer of thickness $(\nu/\omega)^{\frac{1}{2}}$ at the cylinder. We assume that this boundary layer is thin compared to the radius of the cylinder. We define dimensionless variables η, ζ and τ by

$$\eta = \frac{r-R}{(2\nu/\omega)^{\frac{1}{2}}}, \quad \zeta = \frac{z}{(2\nu/\omega)^{\frac{1}{2}}}, \quad \tau = \omega t, \tag{2.1 a, b, c}$$

where (r, θ, z) are cylindrical polar co-ordinates with $r = 0$ corresponding to the axis of the cylinder. If we neglect terms of order $(\nu/\omega)^{\frac{1}{2}} R^{-1}$ in the azimuthal momentum equation we can easily show that for arbitrary values of $\Delta\omega/R$ the basic flow driven by the cylinder is

$$\mathbf{v} = \Delta\omega(0, \mathscr{V}(\eta, \tau), 0), \tag{2.2}$$

where

$$\mathscr{V} = \tfrac{1}{2}\{\exp[i\tau - (1+i)\eta] + \text{c.c.}\}. \tag{2.3}$$

Here c.c. denotes 'complex conjugate'.

Suppose now that the basic flow is perturbed such that the new velocity field is $((2\nu\omega)^{\frac{1}{2}} u, \Delta\omega(\mathscr{V}+v), (2\nu\omega)^{\frac{1}{2}} w)$, where u, v and w are independent of the polar angle θ. We can show from the momentum and continuity equations that u, v, and w satisfy

$$\left(\nabla^2 - 2\frac{\partial}{\partial\tau}\right)\nabla^2 u + T\mathscr{V}\frac{\partial^2 v}{\partial\zeta^2} = -\tfrac{1}{2}T\frac{\partial^2 v^2}{\partial\zeta^2} + \frac{\partial^2 Q_1}{\partial\zeta^2} - \frac{\partial^2 Q_2}{\partial\eta\,\partial\zeta}, \tag{2.4a}$$

$$\left(\nabla^2 - 2\frac{\partial}{\partial\tau}\right)v - 2u\frac{\partial\mathscr{V}}{\partial\eta} = Q_3, \quad \frac{\partial u}{\partial\eta} + \frac{\partial w}{\partial\zeta} = 0, \tag{2.4b, c}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2}, \quad Q_1 = 2\left(u\frac{\partial u}{\partial\eta} + w\frac{\partial u}{\partial\zeta}\right), \quad Q_2 = 2\left(u\frac{\partial w}{\partial\eta} - w\frac{\partial u}{\partial\eta}\right), \quad Q_3 = 2\left(u\frac{\partial v}{\partial\eta} + w\frac{\partial v}{\partial\zeta}\right),$$

whilst T is the Taylor number defined by

$$T = \frac{2\Delta^2}{R}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}. \tag{2.5}$$

We note that in the derivation of (2.4) we have neglected terms of order $(\nu/\omega)^{\frac{1}{2}} R^{-1}$. The differential problem for (u, v, w) is completely specified by stipulating that the velocity components should satisfy the no-slip conditions and vanish outside the boundary layer. Thus we require that

$$u = v = w = 0, \quad \eta = 0, \quad u, v, w \to 0, \quad \eta \to \infty. \tag{2.6 a, b}$$

In I the linearized forms of (2.4) were solved subject to (2.6) in the case when u, v, and w were periodic in ζ with wavelength $2\pi/k$. The resulting eigenvalue problem was investigated and the neutral curve $T = T(k)$ found. The neutral curve was found to be of the usual parabolic shape typical of centrifugal and convective instabilities. The most dangerous disturbance has axial wavenumber $k = a = 0{\cdot}86$ and becomes unstable for $T > 232{\cdot}5$.

The nonlinear development of this particular mode was discussed in II, where it was found that a stable finite-amplitude Taylor vortex flow with wavenumber 0·86 bifurcates supercritically at $T = T_1 = 232{\cdot}5$. Using a multiple scale expansion the bifurcating solution was obtained as a power series in $(T - 232{\cdot}5)^{\frac{1}{2}}$ for $T > 232{\cdot}5$. The equilibrium flow constructed in this way can be expressed in the form $(u, v, w) = \mathbf{u}_E$, where

$$\mathbf{u}_E = (0, V_M(\eta, \tau), 0) + \sum_{n=1}^{\infty}\{(u_n(\eta, \tau), v_n(\eta, \tau), w_n(\eta, \tau))\, e^{ina\zeta} + \text{c.c.}\}. \tag{2.7}$$

The functions V_M, u_n, etc. depend on η and are periodic in τ with period 2π. Moreover the azimuthal velocity components in (2.7) have no steady terms in their Fourier decomposition.

We now suppose that when the Taylor number is increased further the equilibrium flow is still of the form given by (2.7). This assumption is entirely consistent with the experimentally observed flow before the second instability but for $(T-232{\cdot}5)^{\frac{1}{2}}$ no longer small it might be necessary to compute $\mathbf{u}_E$ numerically.

Suppose that we now perturb this equilibrium flow such that (u, v, w) is given by

$$(u, v, w) = \mathbf{u}_E + [(U, V, W)\, e^{\frac{1}{2}ia\zeta} + \text{c.c.}], \tag{2.8}$$

where U, V, and W are small and depend only on η and τ. It is known from I that with $\mathbf{u}_E = 0$ such a disturbance is neutrally stable for $T = T_2 = 328$. If we substitute for (u, v, w) from (2.8) into (2.4) and equate terms proportional to $\exp(\frac{1}{2}ia\zeta)$ we obtain

$$\left(\frac{\partial^2}{\partial\eta^2} - \tfrac{1}{4}a^2 - 2\frac{\partial}{\partial\tau}\right)\left(\frac{\partial^2}{\partial\eta^2} - \tfrac{1}{4}a^2\right) U - \tfrac{1}{4}a^2 T\mathscr{V} V$$
$$= \tfrac{1}{4}a^2 T v_1 V^* - \tfrac{1}{4}Ta^2 v_M V - \tfrac{1}{2}a^2(U^*_\eta u_1 + U^* u_{1\eta} + iau_1 W^* - \tfrac{1}{2}iaw_1 U^*)$$
$$- ia(U^* w_{1\eta} + u_1 W^*_\eta - U^*_\eta w_1 - u_{1\eta} W^*]\,\eta, \tag{2.9a}$$

$$\left(\frac{\partial^2}{\partial\eta^2} - \tfrac{1}{4}a^2 - 2\frac{\partial}{\partial\tau}\right) V - 2U\frac{\partial\mathscr{V}}{\partial\eta} = 2\{u_1 V^*_\eta + U^* v_{1\eta} + ia(W^* v_1 - \tfrac{1}{2}w_1 V^*)\}, \tag{2.9b}$$

$$\frac{\partial U}{\partial\eta} + \tfrac{1}{2}iaW = 0. \tag{2.9c}$$

The boundary conditions needed to completely specify the problem for (U, V, W) are

$$U = V = W = 0, \quad \eta = 0, \quad U, V, W \to 0, \quad \eta \to \infty. \tag{2.10a, b}$$

We note that in (2.9a, b) the functions $\mathscr{V}$, u_1, v_1 and w_1 are assumed to be known. Moreover these functions are all periodic in τ so that, at least in principle, we can solve (2.9), (2.10) by first writing

$$U = e^{\Omega\tau} \sum_{-\infty}^{\infty} U_n e^{in\tau}, \quad \text{etc.}$$

We then substitute these expansions into (2.9), (2.10) and equate like powers of $\exp(i\tau)$. The resulting infinite set of coupled linear differential equations constitutes an eigenvalue problem, $\Omega = \Omega_T$. We are interested in finding the value of T where $\text{Re}(\Omega)$ vanishes. We shall now try and approximate this eigenrelation by perturbation methods.

3. An approximate solution of the eigenvalue problem for the subharmonic mode

We shall now discuss how the rather formidable eigenvalue problem associated with (2.9) and (2.10) can be investigated using perturbation methods. We recall that the fundamental mode ($k = a$) and its subharmonic ($k = \frac{1}{2}a$) are neutrally stable for $T = T_1 = 232{\cdot}5$ and $T = T_2 = 328$, respectively. We define the quantity δ by

$$\delta = \frac{T_2 - T_1}{T_2}, \tag{3.1}$$

so that δ is a measure of the closeness of these eigenvalues. In fact δ has the numerical value 0·29 but we assume that this value is sufficiently small for a perturbation solution of (2.9) to be developed in powers of δ. The expansion which we will set up will be clearly applicable to other problems where almost coincident eigenvalues occur and correspond to Fourier modes one of which is the first harmonic of the other. However, the validity of such an approach in connection with the stability problem of interest here can only be checked by a full numerical investigation of the partial differential system specified by (2.9) and (2.10).

It is worth pointing out at this stage the similarity of the present approach to that used by Davey, DiPrima & Stuart (1968) who investigated the stability of steady Taylor vortices to wavy modes. In the latter paper the quantity corresponding to δ is the separation of the linear critical Taylor numbers for axisymmetric and non-axisymmetric disturbances. This quantity is again finite but Davey, DiPrima & Stuart treat it as a small parameter and obtain results in good agreement with experimental observations.

It is clear that before solving the differential system given by (2.9) and (2.10) we must find some way of approximating the functions u_1, v_1, w_1 and V_M which appear as coefficients in (2.9). We shall do this by developing perturbation expansions for these functions in powers of δ. In fact it is advantageous to return instead to the differential equations given by (2.4) and set up a perturbation expansion of these equations which includes both the equilibrium perturbation $\mathbf{u}_E$ and the subharmonic disturbance.

We first expand the Taylor number in the form

$$T = T_1 + \lambda\delta^2 + \dots \tag{3.2}$$

and note that since from (3.1) we can write $T_1 = T_2 - \delta T_1 + O(\delta^2)$; this can also be written in the form

$$T = T_2 - T_1\delta + O(\delta)^2. \tag{3.3}$$

Thus we are interested in the development of the subharmonic mode in a Taylor-number regime close to the value of T $(= T_1)$ at which the fundamental mode bifurcates supercritically. In such a regime the damping rate of the subharmonic mode in the absence of the fundamental is $O(\delta)$. The amplitude of the first mode which bifurcates at $T = T_1$ is $O(\delta)$ in a δ^2-neighbourhood of T_1 so that the nonlinear interaction of this mode with the subharmonic leads to a growth rate of order δ for the latter mode. We define two slow time variables τ_1 and τ_2 by

$$\tau_1 = \delta\tau, \quad \tau_2 = \delta^2\tau, \tag{3.4 a, b}$$

and introduce a parameter ϵ representing the size of the linear subharmonic mode. We then expand u in the form

$$\begin{aligned} u = {} & \tfrac{1}{2}\delta(f_{10}^{1}\, e^{ia\zeta} + f_{10}^{1*}\, e^{-ia\zeta}) + \delta^2(f_{01}^{1} + \tfrac{1}{2}f_{20}^{1}\, e^{2ia\zeta} + \tfrac{1}{2}f_{20}^{1*}\, e^{-2ia\zeta}) \\ & + \tfrac{1}{2}\delta^3(f_{11}^{1}\, e^{ia\zeta} + f_{11}^{1*}\, e^{-ia\zeta} + f_{30}^{0}\, e^{3ia\zeta} + f_{30}^{0*}\, e^{-3ia\zeta}) \\ & + \tfrac{1}{2}\epsilon(f_{10}^{2}\, e^{\frac{1}{2}ia\zeta} + f_{10}^{2*}\, e^{-\frac{1}{2}ia\zeta}) + \tfrac{1}{2}\epsilon\delta(f_{11}^{2}\, e^{\frac{1}{2}ia\zeta} + f_{12}^{2*}\, e^{-\frac{1}{2}ia\zeta}) \\ & + \tfrac{1}{2}\epsilon\delta\{f_{13}\, e^{\frac{3}{2}ia\zeta} + \dots\} + O(\delta^4, \epsilon^2, \epsilon\delta^2), \end{aligned} \tag{3.5}$$

together with a similar expansion for the azimuthal velocity component v in terms of the functions $g^k_{nm}(\eta, \tau, \tau_1, \tau_2)$. The fast time dependence of the coefficients in these expansions is taken to be periodic so that we can write

$$f^k_{nm} = \sum_{r=-\infty}^{\infty} f^k_{nm,2r}(\eta, \tau_1, \tau_2)\, e^{2ir\tau}, \tag{3.6a}$$

$$g^k_{nm} = \sum_{r=-\infty}^{\infty} g^k_{nm,2r-1}(\eta, \tau_1, \tau_2)\, e^{(2m-1)i\tau}. \tag{3.6b}$$

The only restriction which we impose on ϵ is that $\epsilon \ll \delta^{\frac{3}{2}}$ and the reason for this condition will soon become apparent. We must now substitute the expansions for T, u, and v given above into (2.4) and solve successively the systems of equations obtained by equating in turn terms proportional to δ, δ^2, δ^3, ϵ and $\epsilon\delta$. Each of these partial differential equations is solved such that the no-slip condition is satisfied at the cylinder and there is no motion away from the Stokes layer. It is of interest to note that if the expansion (3.6a) had been in terms of even powers of $\exp(i\tau)$ then the motion would not be confined to the Stokes layer because of steady streaming effects.

At orders δ and ϵ we find that the solutions of the appropriate partial differential systems can be written in the form

$$(f^1_{10,2r}, g^1_{10,2r-1}) = A(\tau_1, \tau_2)\,(u^{(c)}_{2r}(\eta), v^{(c)}_{2r-1}(\eta)), \tag{3.7a}$$

$$(f^2_{10,2r}, g^2_{10,2r-1}) = B(\tau_1, \tau_2)\,(\hat{u}^{(c)}_{2r}(\eta), \hat{v}^{(c)}_{2r-1}(\eta)), \tag{3.7b}$$

where A and B are amplitude functions to be determined at higher order. The functions $u^{(c)}_{2r}(\eta)$, $v^{(c)}_{2r-1}(\eta)$ are given explicitly in I and are evaluated at $T = 232{\cdot}5$ and $k = 0{\cdot}86$ whilst $\hat{u}^{(c)}_{2r}(\eta)$, $\hat{v}^{(c)}_{2r-1}(\eta)$ are the same functions but evaluated at $T = 328$, $k = 0{\cdot}43$.

At order δ^2 we find that A is in fact a function of τ_2 only. At this order we also determine the mean flow correction and the first harmonic terms in the expansion of the velocity field. We obtain

$$(f^1_{01,2r}, g^1_{01,2r-1}) = |A|^2\,(0, G_{01,2r-1}(\eta)), \tag{3.8a}$$

$$(f^1_{20,2r}, g^1_{20,2r-1}) = A^2(F_{20,2r}(\eta), G_{20,2r-1}(\eta)), \tag{3.8b}$$

where $G_{01,2r-1}$, $F_{20,2r}$, $G_{20,2r-1}$ are determined by (3.11) and (3.8), (3.9) of II respectively.

At order δ^3 we equate terms proportional to $\exp(ia\zeta)$ to obtain an inhomogeneous differential system which only has a solution if an orthogonality condition is satisfied. This condition can be written in the form

$$\frac{dA}{d\tau_2} = -a_1 A\lambda - a_2 A|A|^2, \tag{3.9}$$

where a_1 and a_2 are given by (3.15) and (3.16) of II respectively. In a similar manner we find that if the order-$\epsilon\delta$ differential system obtained by equating terms proportional to $\exp(\frac{1}{2}ia\zeta)$ is to have a solution then

$$\frac{\partial B}{\partial \tau_1} = b_1 T_1 B - b_2 AB^*, \tag{3.10}$$

where $*$ denotes complex conjugate and b_1 is given by (3.15) of II but with a replaced by $\frac{1}{2}a$ and the eigenfunctions and adjoint functions now evaluated at $T = 328, k = 0{\cdot}43$. The constant b_2 is given by (3.16) of II but with the following changes: (1) the mean flow terms $G_{01,2r-1}$ are now set equal to zero; (2) the linear eigenfunctions $u^{(c)}_{2r}$, $v^{(c)}_{2r-1}$ are replaced by $\hat{u}^{(c)}_{2r}, \hat{v}^{(c)}_{2r-1}$; (3) the adjoint eigenfunctions F^+_{2r}, G^+_{2r-1} are replaced by the corresponding functions evaluated at $k = 0{\cdot}43$, $T = 328$; (4) the functions $F_{20,2r}$, $G_{20,2r-1}$ are replaced by $u^{(c)}_{2r}, v^{(c)}_{2r-1}$ respectively; (5) the wavenumber a is replaced by $\frac{1}{2}a$ and T_0 by T_2.

The constants a_1, a_2, b_1 and b_2 have the following numerical values:

$$a_1 = -0{\cdot}0023, \quad a_2 = 0{\cdot}038, \quad b_1 = -0{\cdot}00078, \quad b_2 = 0{\cdot}041.$$

The author is indebted to Dr P. W. Duck who carried out the computations required to evaluate these constants.

It follows from (3.9), (3.10) that a possible neutrally stable equilibrium flow for $\lambda < 0$ is $A = B = 0$. For λ positive a finite-amplitude motion with $|A|^2 = -a_1\lambda a_2^{-1}$, $B = 0$ bifurcates from the zero solution, which itself becomes unstable. This motion corresponds to that which is experimentally observed before the second mode appears. Suppose that this flow is perturbed by writing

$$B = b\,e^{\sigma\tau}, \quad A = (a_1\lambda a_2^{-i})^{\frac{1}{2}}\,e^{i\phi}, \tag{3.11 a,b}$$

where ϕ is the phase of the fundamental mode. If we substitute the above forms into (3.10) then by taking real and imaginary parts we obtain two simultaneous equations for the real and imaginary parts of b. If these equations are to have a consistent solution then σ is found to be given by

$$\sigma = b_1 T_1 \pm (-a_1\lambda a_2^{-1} b_2^2)^{\frac{1}{2}}. \tag{3.12}$$

Thus we see that one of the two possible values of σ becomes positive when

$$\lambda = \lambda_c = b_1^2 T_1^2 a_2 b_2^{-2} a_1^{-1},$$

and λ_c has the numerical value 323. Thus at this value of λ the bifurcating solution with wavenumber 0·86 becomes linearly unstable to the subharmonic with wavenumber 0·43. We identify this occurrence with the experimentally observed 'doubling up' of vortices described in I. We are encouraged to believe that this is in fact the case because, if we use (3.2), then we see that this critical value of λ corresponds to a Taylor number of 260 which is in remarkable agreement with the experimentally obtained value of 262 given in I.

The fact that this Taylor number differs from T_2 by about $\frac{1}{2}(T_2 - T_1)$ raises the question as to whether we were justified in expanding T in the form (3.2). We further note that when $\lambda = \lambda_c$ the equilibrium amplitude of the first mode is given approximately by $|A| \sim 4$. The closeness of our theoretical prediction of the second critical Taylor number encourages us to believe that the ordering of terms which we have carried out is sensible. However, we recognize that a confirmation of the reliability of this ordering can only be obtained by the numerical solution of (2.9).

4. Conclusions

We have shown that the Taylor vortex flow which can exist in a Stokes layer on a torsionally oscillating circular cylinder becomes unstable to a subharmonic mode of instability when a critical value of the Taylor number is exceeded. The theory which we have given shows that, on the basis of linear stability theory, the subharmonic mode will grow exponentially in time for $T > 260$. Thus the approximate theory which we have given leads to excellent agreement with the experimental value $T = 262$

given in I. The corresponding value found by Donnelly & Park (1980) for the largest value of ratio of cylinder radius to Stokes-layer thickness investigated is $T = 257$. This remarkable agreement between theory and experiment is perhaps fortuitous but, even if this is the case, we believe that the present approach indicates the mechanism by which the fundamental mode interacts with the subharmonic to destabilize the latter mode. We do not investigate here the possibility of the existence of an equilibrium flow including the fundamental and the subharmonic. However, the (available) experimental results suggest that no such flow exists. Moreover it is observed experimentally that after the initial doubling up the flow becomes θ dependent and a steady velocity component around the cylinder is generated. Thus we expect that an adequate description of the experimental results requires a nonlinear theory which allows for the existence of the fundamental, subharmonic, and the θ-dependent modes discussed by Seminara (1979). We do not pursue this question further here.

The subharmonic destabilization mechanism which we have discussed in § 3 is clearly not restricted to Stokes-layer instabilities. The expansion procedure used in § 3 is clearly appropriate to any stability problem where an equilibrium flow with wavenumber a bifurcates supercritically from a Reynolds number R_1 (or Rayleigh number) close to a second Reynolds number R_2 ($> R_1$) at which the mode with wavenumber $\frac{1}{2}a$ is linearly unstable. Our analysis shows that, if δ is a measure of the closeness of these Reynolds numbers, then the bifurcating fundamental mode becomes linearly unstable to the subharmonic at a Reynolds number differing from R_1 by $O(\delta^2 R_1)$. We note that the expansion procedure used will be formally valid only in the limit $\delta \to 0$. However, in any particular problem it is possible that results based on such a procedure but with δ finite are meaningful. The validity of such expansions can of course be checked by proceeding to higher order and/or investigating the problem in question numerically.

Kelly (1968) has discussed a subharmonic destabilization mechanism in connection with shear flow instabilities. In that paper Kelly assumed a basic flow having a component proportional to the fundamental mode of instability. The linear stability of this flow to a subharmonic mode was then investigated. The size of the fundamental mode required to produce the linear growth of the subharmonic was determined. In our problem we are not in a position to 'choose' the size of the fundamental since we insist that the fundamental is a solution of the equations of motion.

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REFERENCES

Davey, A., DiPrima, R. C. & Stuart, J. T. 1968 *J. Fluid Mech.* **31**, 17.
Donnelly, R. J. & Park, K. 1980 (to be published).
Duck, P. W. 1979 *Quart. J. Mech. Appl. Math.* **32**, 73.
Kelly, R. E. 1968 *J. Fluid Mech.* **31**, 17.
Seminara, G. 1979 *Z. angew. Mech. Phys.* **30**, 615.
Seminara, G. & Hall, P. 1976 *Proc. Roy. Soc.* A **350**, 299.
Seminara, S. & Hall, P. 1977 *Proc. Roy. Soc.* A **354**, 119.